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p-mechanics as a physical theory: an introduction

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Abstract
This paper provides an introduction to *p*-mechanics, which is a consistent physical theory suitable for a simultaneous description of classical and quantum mechanics. *p*-mechanics naturally provides a common ground for several different approaches to quantization (geometric, Weyl, coherent states, Berezin, deformation, Moyal, etc) and has a potential for expansions into field and string theories. The backbone of *p*-mechanics is solely the representation theory of the Heisenberg group.

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1. Introduction

This paper describes how classical and quantum mechanics are naturally combined within a construction based on the Heisenberg group $\mathbb{H}^n$ and the complete set of its unitary representations. There is a dynamic equation (4.9) on $\mathbb{H}^n$ which generates both Heisenberg (4.10) and Hamilton (4.11) equations and corresponding classical and quantum dynamics. The standard assumption that observables constitute an algebra, which is discussed in [24, 26] and elsewhere, is not necessary for setting up a valid quantization scheme.

The outline of the paper is as follows. In the next section we recall the representation theory of the Heisenberg group based on the *orbit method* of Kirillov [21] utilizing Fock–Segal–Bargmann spaces [11, 15]. We emphasize the existence *and* usability of the family of one-dimensional representations: they play for classical mechanics exactly the same role as infinite dimensional representations do for quantum mechanics. In section 3 we introduce the concept of observables in *p*-mechanics and describe their relations with quantum and classical observables. These links are provided by the representations of the Heisenberg group and *wavelet* transforms. In section 4 we study *p*-mechanical brackets and the associated dynamic equation together with its classical and quantum representations. In conclusion we derive the symplectic invariance of dynamics from automorphisms of $\mathbb{H}^n$.

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The notion of physical states in p-mechanics is introduced in subsequent publications [6, 7]; the p-mechanical approach to quantized fields is sketched in [28] with some further papers to follow.

2. The Heisenberg group and its representations

We start from the representation theory of the Heisenberg group $\mathbb{H}^n$ based on the *orbit method* of Kirillov. Analysis of the unitary dual of $\mathbb{H}^n$ in section 2.2 suggests that the family of one-dimensional representations of $\mathbb{H}^n$ forms the phase space of a classical system. Infinite dimensional representations in the Fock type space are described in section 2.3.

2.1. Representations $\mathbb{H}^n$ and method of orbit

Let (s, x, y), where $x, y \in \mathbb{R}^n$ and $s \in \mathbb{R}$, be an element of the Heisenberg group $\mathbb{H}^n$ [11, 15]. We assign physical units to coordinates on $\mathbb{H}^n$. Let M be the unit of mass, L that of length and T that of time, we then adopt the following.

Convention 2.1.

(1) Only physical quantities of the *same dimension* can be added or subtracted.
(2) Therefore mathematical functions, e.g. $\exp(u) = 1 + u + u^2/2! + \cdots$ or $\sin(u)$, can be naturally constructed out of a dimensionless number u only. Thus Fourier dual variables, say x and q, should posses reciprocal dimensions because they have to form the expression e^{ixq}.
(3) We assign to x and y components of (s, x, y) physical units $1/L$ and $T/(LM)$ respectively.

Convention 2.1.3 is the only *a priori* assumption which we made about physical dimensions and it will be justified *a posteriori* as follows. From 2.1.2 we need dimensionless products qx and py in order to get the exponent in (2.15), where q and p represent the classical coordinates and momenta (in accordance with the main observation of p-mechanics). All other dimensions will be assigned strictly in agreement with the conventions 2.1.1 and 2.1.2.

The group law on $\mathbb{H}^n$ is given as follows:

$$(s, x, y) * (s', x', y') = \left(s + s' + \tfrac{1}{2}\omega(x, y; x', y'), x + x', y + y'\right) \tag{2.1}$$

where the non-commutativity is solely due to ω—the *symplectic form* [2, section 37] on the Euclidean space $\mathbb{R}^{2n}$:

$$\omega(x, y; x', y') = xy' - x'y. \tag{2.2}$$

Consequently the parameter s should be measured in $T/(L^2M)$—the product of units of x and y. The Lie algebra $\mathfrak{h}^n$ of $\mathbb{H}^n$ is spanned by the basis $S, X_j, Y_j, j = 1, \ldots, n$, which may be represented by either left- or right-invariant vector fields on $\mathbb{H}^n$:

$$S^{l(r)} = \pm\frac{\partial}{\partial s} \qquad X_j^{l(r)} = \pm\frac{\partial}{\partial x_j} - \frac{y_j}{2}\frac{\partial}{\partial s} \qquad Y_j^{l(r)} = \pm\frac{\partial}{\partial y_j} + \frac{x_j}{2}\frac{\partial}{\partial s}. \tag{2.3}$$

These fields satisfy the Heisenberg *commutator relations* expressed through the Kronecker delta $\delta_{i,j}$ as follows:

$$\left[X_i^{l(r)}, Y_j^{l(r)}\right] = \delta_{i,j} S^{l(r)} \tag{2.4}$$

and all other commutators (including those between a left and a right field) vanish. Units to measure $S^{l(r)}$, $X_j^{l(r)}$ and $Y_j^{l(r)}$ are inverse to s, x, y—i.e. L^2M/T, L and LM/T respectively—which are obviously compatible with (2.4).

The exponential map $\exp : \mathfrak{h}^n \to \mathbb{H}^n$ which obeys the multiplication (2.1) and Heisenberg commutators (2.4) is provided by

$$\exp : sS + \sum_{j=1}^{n}(x_j X_j + y_j Y_j) \mapsto (s, x_1, \ldots, x_n, y_1, \ldots, y_n).$$

The composition of the exponential map with representations (2.3) of $\mathfrak{h}^n$ by the left (right)-invariant vector fields produces the right (left) regular representation $\lambda_{r(l)}$ of $\mathbb{H}^n$ by right (left) shifts. Linearized [19, section 7.1] to $L_2(\mathbb{H}^n)$ they are

$$\lambda_r(g) : f(h) \mapsto f(hg) \qquad \lambda_l(g) : f(h) \mapsto f(g^{-1}h) \qquad \text{where} \quad f(h) \in L_2(\mathbb{H}^n). \tag{2.5}$$

As any group $\mathbb{H}^n$ acts on itself by the conjugation automorphisms $\mathrm{A}(g)h = g^{-1}hg$, which fix the unit $e \in \mathbb{H}^n$. The differential $\mathrm{Ad} : \mathfrak{h}^n \to \mathfrak{h}^n$ of A at e is a linear map which can be differentiated again to the representation Ad of the Lie algebra $\mathfrak{h}^n$ by the commutator: $\mathrm{ad}(A) : B \mapsto [B, A]$. The adjoint space $\mathfrak{h}^*_n$ of the Lie algebra $\mathfrak{h}^n$ can be realized by the left-invariant first order differential forms on $\mathbb{H}^n$. By the duality between $\mathfrak{h}^n$ and $\mathfrak{h}^*_n$ the map Ad generates the *co-adjoint representation* [19, section 15.1] $\mathrm{Ad}^* : \mathfrak{h}^*_n \to \mathfrak{h}^*_n$:

$$\mathrm{Ad}^*(s, x, y) : (h, q, p) \mapsto (h, q + hy, p - hx) \qquad \text{where} \quad (s, x, y) \in \mathbb{H}^n \tag{2.6}$$

and $(h, q, p) \in \mathfrak{h}^*_n$ in biorthonormal coordinates to the exponential ones on $\mathfrak{h}^n$. These coordinates h, q, p should have units of an *action* ML^2/T, *coordinates* L and *momenta* LM/T correspondingly. Again nothing in (2.6) violates the convention 2.1.

There are two types of orbits for Ad^* (2.6): isomorphic to Euclidean spaces $\mathbb{R}^{2n}$ and single points:

$$\mathcal{O}_h = \{(h, q, p) : \text{for a fixed } h \neq 0 \text{ and all } (q, p) \in \mathbb{R}^{2n}\} \tag{2.7}$$

$$\mathcal{O}_{(q,p)} = \{(0, q, p) : \text{for a fixed } (q, p) \in \mathbb{R}^{2n}\}. \tag{2.8}$$

The *orbit method* of Kirillov [19, section 15, 21] starts from the observation that the above orbits parametrize all irreducible unitary representations of $\mathbb{H}^n$. All representations are *induced* [19, section 13] by the character $\chi_h(s, 0, 0) = \mathrm{e}^{2\pi \mathrm{i} hs}$ of the centre of $\mathbb{H}^n$ generated by $(h, 0, 0) \in \mathfrak{h}^*_n$ and shifts (2.6) from the 'left-hand side' (i.e. by g^{-1}) on orbits. Using [19, section 13.2, prob. 5] we get a neat formula, which (unlike some others in the literature, e.g. [33, chapter 1, (2.23)]) obeys convention 2.1 for all physical units:

$$\rho_h(s, x, y) : f_h(q, p) \mapsto \mathrm{e}^{-2\pi \mathrm{i}(hs + qx + py)} f_h\left(q - \frac{h}{2}y, p + \frac{h}{2}x\right). \tag{2.9}$$

Exactly the same formula is obtained if we apply the Fourier transform $\hat{\ } : L_2(\mathbb{H}^n) \to L_2(\mathfrak{h}^*_n)$ given by

$$\hat{\phi}(F) = \int_{\mathfrak{h}^n} \phi(\exp X)\, \mathrm{e}^{-2\pi \mathrm{i}\langle X, F\rangle}\, \mathrm{d}X \qquad \text{where} \quad X \in \mathfrak{h}^n \quad F \in \mathfrak{h}^*_n \tag{2.10}$$

to the left regular action (2.5); see [21, section 2.3] for relations of the Fourier transform (2.10) and the orbit method.

The derived representation $\mathrm{d}\rho_h$ of the Lie algebra $\mathfrak{h}^n$ defined on the vector fields (2.3) is

$$\mathrm{d}\rho_h(S) = -2\pi \mathrm{i} h I \qquad \mathrm{d}\rho_h(X_j) = \frac{h}{2}\partial_{p_j} - 2\pi \mathrm{i} q_j I \qquad \mathrm{d}\rho_h(Y_j) = -\frac{h}{2}\partial_{q_j} - 2\pi \mathrm{i} p_j I \tag{2.11}$$

which clearly represents the commutation rules (2.4). The representation ρ_h (2.9) is reducible on the whole of $L_2(\mathcal{O}_h)$ as can be seen from the existence of the set of 'right-invariant', i.e. commuting with (2.11), differential operators:

$$\mathrm{d}\rho_h^r(S) = 2\pi \mathrm{i} h I \qquad \mathrm{d}\rho_h^r(X_j) = -\frac{h}{2}\partial_{p_j} - 2\pi \mathrm{i} q_j I \qquad \mathrm{d}\rho_h^r(Y_j) = \frac{h}{2}\partial_{q_j} - 2\pi \mathrm{i} p_j I \tag{2.12}$$

which also represent the commutation rules (2.4).

To obtain an irreducible representation defined by (2.9), we need to restrict it to a subspace of $L_2(\mathcal{O}_h)$ where operators (2.12) act as scalars, e.g. use a *polarization* from the *geometric quantization* [35]. For $h > 0$ consider the vector field $-X_j + \mathrm{i}c_\mathrm{i}Y_j$ from the complexification of $\mathfrak{h}^n$, where the constant c_i has the dimension T/M in order to satisfy convention 2.1, the numerical value of c_i in given units can be assumed 1. We introduce operators D_h^j, $1 \leqslant j \leqslant n$, representing vectors $-X_j + \mathrm{i}c_\mathrm{i}Y_j$:

$$D_h^j = \mathrm{d}\rho_h^r(-X_j + \mathrm{i}c_\mathrm{i}Y_j) = \frac{h}{2}\left(\partial_{p_j} + c_\mathrm{i}\mathrm{i}\partial_{q_j}\right) + 2\pi(c_\mathrm{i}p_j + \mathrm{i}q_j)I = h\partial_{\bar{z}_j} + 2\pi z_j I \tag{2.13}$$

where $z_j = c_\mathrm{i}p_j + \mathrm{i}q_j$. For $h < 0$ we define $D_h^j = \mathrm{d}\rho_h^r(-c_\mathrm{i}Y_j + \mathrm{i}X_j)$. Operators (2.13) are used to give the following classical result in terms of orbits.

Theorem 2.2 (Stone–von Neumann, cf [11, chapter 1, section 5], [19, section 18.4]). *All unitary irreducible representations of $\mathbb{H}^n$ are parametrized up to equivalence by two classes of orbits (2.7) and (2.8) of co-adjoint representation (2.6) in $\mathfrak{h}_n^*$:*

(1) The infinite dimensional representations by transformation ρ_h (2.9) for $h \neq 0$ in Fock [11, 15] space $F_2(\mathcal{O}_h) \subset L_2(\mathcal{O}_h)$ of null solutions to the operators D_h^j (2.13),

$$F_2(\mathcal{O}_h) = \left\{ f_h(q,p) \in L_2(\mathcal{O}_h) \,\middle|\, D_h^j f_h = 0, 1 \leqslant j \leqslant n \right\}. \tag{2.14}$$

(2) The one-dimensional representations as multiplication by a constant on $\mathbb{C} = L_2(\mathcal{O}_{(q,p)})$ which drop out from (2.9) for $h = 0$,

$$\rho_{(q,p)}(s,x,y) : c \mapsto \mathrm{e}^{-2\pi\mathrm{i}(qx+py)}c \tag{2.15}$$

with the corresponding derived representation

$$\mathrm{d}\rho_{(q,p)}(S) = 0 \qquad \mathrm{d}\rho_{(q,p)}(X_j) = -2\pi\mathrm{i}q_j \qquad \mathrm{d}\rho_{(q,p)}(Y_j) = -2\pi\mathrm{i}p_j. \tag{2.16}$$

2.2. Structure and topology of the unitary dual of $\mathbb{H}^n$

The structure of the unitary dual object to $\mathbb{H}^n$—the collection of all different classes of unitary irreducible representations—as it appears from the method of orbits is illustrated by figure 1, cf [20, chapter 7, figures 6 and 7]. The adjoint space $\mathfrak{h}_n^*$ is sliced into 'horizontal' hyperplanes. A plane with a parameter $h \neq 0$ forms a single orbit (2.7) and corresponds to a particular class of unitary irreducible representation (2.9). The plane with parameter $h = 0$ is a family of one-point orbits $(0, q, p)$ (2.8), which produces one-dimensional representations (2.15). The topology on the dual object is the factor topology inherited from the adjoint space $\mathfrak{h}_n^*$ under the above identification, see [21, section 2.2].

Example 2.3. A set of representations ρ_h (2.9) with $h \to 0$ is dense in the whole family of one-dimensional representations (2.15), as can be seen either from figure 1 or analytic expressions (2.9) and (2.15) for those representations.

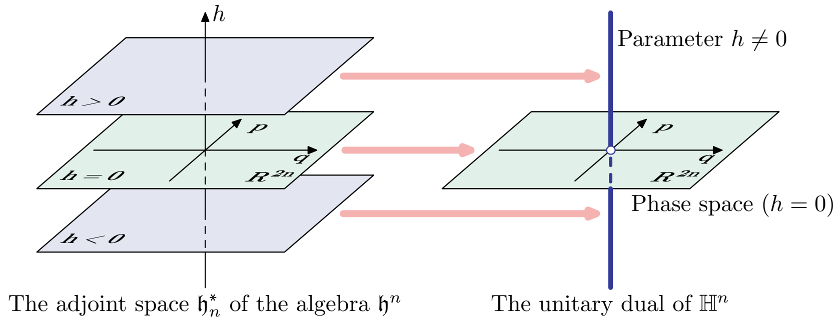

Figure 1. The structure of a unitary dual object to $\mathbb{H}^n$ appearing from the method of orbits. The space $\mathfrak{h}_n^*$ is sliced into 'horizontal' hyperplanes. Planes with $h \neq 0$ form single orbits and correspond to different classes of unitary irreducible representation. The plane $h = 0$ is a family of one-point orbits $(0, q, p)$, which produce one-dimensional representations. The topology on the dual object is the factor topology inherited from the $\mathfrak{h}_n^*$ [21, section 2.2].

Non-commutative representations ρ_h, $h \neq 0$ (2.9) are known to be connected with quantum mechanics [11] from its origin. This explains, for example, the name of the Heisenberg group. In contrast, commutative representations (2.15) are mostly neglected and only mentioned for the sake of completeness in some mathematical formulations of the Stone–von Neumann theorem. The development of *p-mechanics* started [23] from the observation that the union of all representations $\rho_{(q,p)}$, $(q, p) \in \mathbb{R}^{2n}$ naturally acts as the classical *phase space*. The sensibility of the single union

$$\mathcal{O}_0 = \bigcup_{(q,p)\in\mathbb{R}^{2n}} \mathcal{O}_{(q,p)} \tag{2.17}$$

rather than an unrelated set of disconnected orbits manifests itself in several ways:

(1) The topological position of $\mathcal{O}_0$ as the limiting case (cf example 2.3) of quantum mechanics for $h \to 0$ realizes the *correspondence principle* between quantum and classical mechanics.
(2) Symplectic automorphisms of the Heisenberg group (see section 4.3) produce the metaplectic representation in quantum mechanics and *transitively* act by linear symplectomorphisms on the whole set $\mathcal{O}_0 \setminus \{0\}$.
(3) We got the Poisson brackets (4.7) on $\mathcal{O}_0$ from the same source (4.2) that leads to the correct Heisenberg equation in quantum mechanics.

The identification of $\mathcal{O}_0$ with the classical phase space justifies q and p being measured by the units of length and momentum respectively, which supports our choice of units for x and y in convention 2.1.3.

Remark 2.4. Since unitary representations are classified up to a unitary equivalence, one may think that their explicit realizations in particular Hilbert spaces are 'the same'. However, a suitable form of representation can give many technical advantages. The classical illustration is the paper [15], where comparison of the (unitary equivalent!) Schrödinger and Fock representations of $\mathbb{H}^n$ is the principal tool of investigation.

Our form (2.9) of representations of $\mathbb{H}^n$ given in theorem 2.2 has at least the two following advantages, which are rarely combined together:

(1) There is an explicit physical meaning of all entries in (2.9) as will be seen below. In contrast, formula (2.23) in [33, chapter 1] contains terms $\sqrt{h}$ (in our notation), which could hardly be justified from a physical point of view.
(2) The one-dimensional representations (2.15) explicitly correspond to the case $h = 0$ in (2.9). The Schrödinger representation (the most used in quantum mechanics!) is handicapped in this sense: a transition $h \to 0$ from ρ_h in the Schrödinger form to $\rho_{(q,p)}$ requires a long discussion [20, ex. 7.11].

We finish the discussion of the unitary dual of $\mathbb{H}^n$ by a remark about negative values of h. According to the Heisenberg equation (4.10), a negative value of $\hbar$ reverses the flow of time. Thus representations ρ_h with $h < 0$ seem to be suitable for a description of anti-particles. There is explicit (cf figure 1) mirror symmetry between matter and anti-matter through classical mechanics. In this paper however we will consider only the case of $h > 0$.

2.3. Fock spaces $F_2(\mathcal{O}_h)$ and coherent states

Our Fock type spaces (2.14) are not very different [25, ex. 4.3] from the standard Segal–Bargmann spaces.

Definition 2.5 [11, 15]**.** *The Segal–Bargmann space (with a parameter $h > 0$) consists of functions on $\mathbb{C}^n$ which are holomorphic in z, i.e. $\partial_{\bar{z}_j} f(z) = 0$, and square integrable with respect to the measure* $\mathrm{e}^{-2|z|^2/h}\,\mathrm{d}z$ *on* $\mathbb{C}^n$:

$$\int_{\mathbb{C}^n} |f(z)|^2\,\mathrm{e}^{-2|z|^2/h}\,\mathrm{d}z < \infty.$$

Noting the $\partial_{\bar{z}_j}$ component in the operator D_h^j (2.13), we obviously obtain the following proposition.

Proposition 2.6. *A function $f_h(q, p)$ is in $F_2(\mathcal{O}_h)$ (2.14) for $h > 0$ if and only if the function $f_h(z)\,\mathrm{e}^{|z|^2/h}$, $z = p + \mathrm{i}q$, is in the classical Segal–Bargmann space.*

The space $F_2(\mathcal{O}_h)$ can also be described in the language of *coherent states* (also known as *wavelets*, *matrix elements* of representation, *Berezin transform*, etc, see [1, 25]). Since the representation ρ_h is irreducible, any vector v_0 in $F_2(\mathcal{O}_h)$ is *cyclic*, i.e. vectors $\rho_h(g)v_0$ for all $g \in G$ span the whole space $F_2(\mathcal{O}_h)$. Even though all vectors are equally good in principle, some of them are more suitable for particular purposes (cf remark 2.4). For the harmonic oscillator the preferred vector is the dimensionless *vacuum state*:

$$v_0(q, p) = \exp\left(-\frac{2\pi}{h}\left(c_{\mathrm{i}}^{-1}q^2 + c_{\mathrm{i}}p^2\right)\right) \tag{2.18}$$

which corresponds to the minimal level of energy. Here c_{i} as was defined before (2.13) has the dimensionality T/M. One can check directly the validity of both equation (2.14) and convention 2.1 for (2.18), particularly that the exponent is taken from a dimensionless pure number. Note also that $v_0(q, p)$ is destroyed by the *annihilation operators* (cf (2.11) and (2.13)):

$$A_h^j = \mathrm{d}\rho_h(X_j + \mathrm{i}c_{\mathrm{i}}Y_j) = \frac{h}{2}\left(\partial_{p_j} - \mathrm{i}c_{\mathrm{i}}\partial_{q_j}\right) + 2\pi(c_{\mathrm{i}}p_j - \mathrm{i}q_j)I. \tag{2.19}$$

We introduce a dimensionless inner product on $F(\mathcal{O}_h)$ by the formula

$$\langle f_1, f_2\rangle = \left(\frac{4}{h}\right)^n \int_{\mathbb{R}^{2n}} f_1(q, p)\bar{f}_2(q, p)\,\mathrm{d}q\,\mathrm{d}p \tag{2.20}$$

With respect to this product the vacuum vector (2.18) is normalized: $\|v_0\| = 1$. For any observable A the formula

$$\langle Av_0, v_0\rangle = \left(\frac{4}{h}\right)^n \int_{\mathbb{R}^{2n}} Av_0(q,p)\bar{v}_0(q,p)\,\mathrm{d}q\,\mathrm{d}p$$

gives an expectation in units of A since both the vacuum vector $v_0(q,p)$ and the inner product (2.20) are dimensionless. The term h^{-n} in (2.20) not only normalizes the vacuum and fixes the dimensionality of the inner product but is also related to the *Plancherel measure* [11, (1.61)] and [33, chapter 1, theorem 2.6] on the unitary dual of $\mathbb{H}^n$.

Elements $(s,0,0)$ of the centre of $\mathbb{H}^n$ trivially act in the representation ρ_h (2.9) as multiplication by scalars, e.g. any function is a common eigenvector of all operators $\rho_h(s,0,0)$. Thus the essential part [25, definition 2.5] of the operator $\rho_h(s,x,y)$ is determined solely by $(x,y)\in\mathbb{R}^{2n}$. The *coherent states* $v_{(x,y)}(q,p)$ are 'left shifts' of the vacuum vector $v_0(q,p)$ by operators (2.9):

$$\begin{aligned} v_{(x,y)}(q,p) &= \rho_h(0,x,y)v_0(q,p)\\ &= \exp\left(-2\pi\mathrm{i}(qx+py) - \frac{2\pi}{h}\left(c_{\mathrm{i}}^{-1}\left(q-\frac{h}{2}y\right)^2 + c_{\mathrm{i}}\left(p+\frac{h}{2}x\right)^2\right)\right). \end{aligned} \tag{2.21}$$

Now any function from the space $F_2(\mathcal{O}_h)$ can be represented [25, ex. 4.3] as a linear superposition of coherent states:

$$\begin{aligned} f(q,p) = [\mathcal{M}_h\breve{f}](q,p) &= h^n \int_{\mathbb{R}^{2n}} \breve{f}(x,y)v_{(x,y)}(q,p)\,\mathrm{d}x\,\mathrm{d}y\\ &= h^n \int_{\mathbb{R}^{2n}} \breve{f}(x,y)\rho_h(x,y)\,\mathrm{d}x\,\mathrm{d}y\; v_{(0,0)}(q,p) \end{aligned} \tag{2.22}$$

where $\breve{f}(x,y)$ is the *wavelet* (or *coherent state*) *transform* [1, 25] of $f(q,p)$:

$$\begin{aligned} \breve{f}(x,y) = [\mathcal{W}_h f](x,y) &= \langle f, v_{(x,y)}\rangle_{F_2(\mathcal{O}_h)}\\ &= \left(\frac{4}{h}\right)^n \int_{\mathbb{R}^{2n}} f(q,p)\bar{v}_{(x,y)}(q,p)\,\mathrm{d}q\,\mathrm{d}p. \end{aligned} \tag{2.23}$$

Formula (2.22) can be regarded [25] as the *inverse wavelet transform* $\mathcal{M}$ of $\breve{f}(x,y)$. Note that all the above integrals are dimensionless, thus both the wavelet transform and its inverse are measured in the same units.

The straightforward use of the basic formula

$$\int_{-\infty}^{\infty} \exp(-ax^2+bx+c)\,\mathrm{d}x = \sqrt{\frac{\pi}{a}}\exp\left(\frac{b^2}{4a}+c\right) \qquad \text{where} \quad a>0 \tag{2.24}$$

for the wavelet transform (2.22) leads to

$$\breve{v}_0(s,x,y) = \exp 2\pi\left(\mathrm{i}hs - \frac{h}{4}\left(c_{\mathrm{i}}x^2 + c_{\mathrm{i}}^{-1}y^2\right)\right). \tag{2.25}$$

Since [25, proposition 2.6] the wavelet transform $\mathcal{W}_h$ (2.22) intertwines ρ_h (2.9) with the left regular representation λ_l (2.5):

$$\mathcal{W}_h \circ \rho_h(g) = \lambda_l(g)\circ\mathcal{W}_h \qquad \text{for all } g\in\mathbb{H}^n$$

the image of an arbitrary coherent state is

$$\begin{aligned} \breve{v}_{(s',x',y')}(s,x,y) = \exp 2\pi &\left(\mathrm{i}h\left(s-s'-\frac{1}{2}(x'y-xy')\right)\right.\\ &\left. - \frac{h}{4}\left(c_{\mathrm{i}}(x-x')^2 + c_{\mathrm{i}}^{-1}(y-y')^2\right)\right). \end{aligned} \tag{2.26}$$

Needless to say, these functions obey convention 2.1.

We should mention however a problem related to coherent states (2.21): all their 'classical limits' for $h \to 0$ are functions with supports in neighbourhoods of (0, 0). In contrast we may wish they are supported around different classical states (q, p). This difficulty can be resolved through a replacement of the group action of $\mathbb{H}^n$ in (2.21) by the 'shifts' (4.8) generated by the p-mechanical brackets (4.3).

3. *p*-mechanics: statics

We define p-mechanical observables to be convolutions on the Heisenberg group. The next subsection describes their multiplication and commutator as well as quantum and classical representations. The Berezin quantization in the form of wavelet transform is considered in section 3.2. This is developed in section 3.3 into a construction of p-observables out of either quantum or classical ones.

3.1. Observables in p-mechanics, convolutions and commutators

In line with the standard quantum theory, we give the following definition.

Definition 3.1. *Observables in p-mechanics (p-observables) are presented by operators on* $L_2(\mathbb{H}^n)$.

Actually we will need here[2] only operators generated by convolutions on $L_2(\mathbb{H}^n)$. Let $\mathrm{d}g$ be a left-invariant measure [19, section 7.1] on $\mathbb{H}^n$, which coincides with the standard Lebesgue measure on $\mathbb{R}^{2n+1}$ in the exponential coordinates (s, x, y). Then a function k_1 from the linear space $L_1(\mathbb{H}^n, \mathrm{d}g)$ acts on $k_2 \in L_2(\mathbb{H}^n, \mathrm{d}g)$ by the convolution as follows:

$$\begin{aligned}(k_1 * k_2)(g) &= c_h^{n+1} \int_{\mathbb{H}^n} k_1(g_1) k_2\big(g_1^{-1} g\big)\,\mathrm{d}g_1 \\ &= c_h^{n+1} \int_{\mathbb{H}^n} k_1\big(g g_1^{-1}\big) k_2(g_1)\,\mathrm{d}g_1 \end{aligned} \tag{3.1}$$

where the constant c_h is measured in units of the action and can be assumed equal to 1. Then c_h^{n+1} has units inverse to $\mathrm{d}g$. Thus the convolution $k_1 * k_2$ is measured in units that are a product of the units for k_1 and k_2. The composition of two convolution operators K_1 and K_2 with kernels k_1 and k_2 has the kernel defined by the same formula (3.1). Clearly two products $K_1 K_2$ and $K_2 K_1$ could have different values due to non-commutativity of $\mathbb{H}^n$ but are always measured in the same units. Thus we can find out how distinct they are from the difference $K_1 K_2 - K_2 K_1$, which does not violate convention 2.1. This also produces the *inner derivations* D_k of $L_1(\mathbb{H}^n)$ by the *commutator*:

$$\begin{aligned} D_k : f \mapsto [k, f] &= k * f - f * k \\ &= c_h^{n+1} \int_{\mathbb{H}^n} k(g_1)\big(f\big(g_1^{-1} g\big) - f\big(g g_1^{-1}\big)\big)\,\mathrm{d}g_1. \end{aligned} \tag{3.2}$$

Because we only consider observables that are convolutions on $\mathbb{H}^n$, we can extend a unitary representation ρ_h of $\mathbb{H}^n$ to a $*$-representation $L_1(\mathbb{H}^n, \mathrm{d}g)$ by the formula

$$\begin{aligned} [\rho_h(k) f](q, p) &= c_h^{n+1} \int_{\mathbb{H}^n} k(g) \rho_h(g) f(q, p)\,\mathrm{d}g \\ &= c_h^n \int_{\mathbb{R}^{2n}} \left(c_h \int_{\mathbb{R}} k(s, x, y)\,\mathrm{e}^{-2\pi \mathrm{i} h s}\,\mathrm{d}s \right) \mathrm{e}^{-2\pi\mathrm{i}(qx+py)} f\left(q - \frac{h}{2} y, p + \frac{h}{2} x\right) \mathrm{d}x\,\mathrm{d}y. \end{aligned} \tag{3.3}$$

[2] More general operators are in use for a string-like version of p-mechanics, see section 5.2.3.

The last formula in the Schrödinger representation defines for $h \neq 0$ a *pseudodifferential operator* [11, 15, 32] on $L_2(\mathbb{R}^n)$ (2.14), which is known to be *quantum observable* in the Weyl quantization. For representations $\rho_{(q,p)}$ (2.15) an expression analogous to (3.3) defines an operator of multiplication on $\mathcal{O}_0$ (2.17) by the Fourier transform of $k(s, x, y)$:

$$\rho_{(q,p)}(k) = \hat{k}\,(0, q, p) = c_h^{n+1} \int_{\mathbb{H}^n} k(s, x, y)\, \mathrm{e}^{-2\pi \mathrm{i}(qx+py)}\, \mathrm{d}s\, \mathrm{d}x\, \mathrm{d}y \tag{3.4}$$

where the direct $\hat{}$ and inverse $\check{}$ Fourier transforms are defined by the formulae:

$$\hat{f}(v) = \int_{\mathbb{R}^m} f(u)\, \mathrm{e}^{-2\pi \mathrm{i} uv}\, \mathrm{d}u \qquad \text{and} \qquad f(u) = (\hat{f})\check{}\,(u) = \int_{\mathbb{R}^m} \hat{f}(v)\, \mathrm{e}^{2\pi \mathrm{i} vu}\, \mathrm{d}v.$$

For reasons discussed in subsections 2.2 and 4.1 we regard the functions (3.4) on $\mathcal{O}_0$ as *classical observables*. Again both representations $\rho_h(k)$ and $\rho_{(q,p)}k$ are measured in the same units as the function k.

From (3.3) it follows that $\rho_h(k)$ for a fixed $h \neq 0$ depends only on $\hat{k}_s(h, x, y)$, which is the partial Fourier transform $s \to h$ of $k(s, x, y)$. Then the representation of the composition of two convolutions depends only on

$$\begin{aligned}
(k' * k)\hat{}_s &= c_h \int_{\mathbb{R}} \mathrm{e}^{-2\pi \mathrm{i} hs} c_h^{n+1} \int_{\mathbb{H}^n} k'(s', x', y') \\
&\qquad \times k\left(s - s' + \frac{1}{2}(xy' - yx'), x - x', y - y'\right) \mathrm{d}s'\, \mathrm{d}x'\, \mathrm{d}y'\, \mathrm{d}s \\
&= c_h^n \int_{\mathbb{R}^{2n}} \mathrm{e}^{\pi \mathrm{i} h(xy'-yx')} c_h \int_{\mathbb{R}} \mathrm{e}^{-2\pi \mathrm{i} hs'} k'(s', x', y')\, \mathrm{d}s' c_h \int_{\mathbb{R}} \mathrm{e}^{-2\pi \mathrm{i} h(s-s'+\frac{1}{2}(xy'-yx'))} \\
&\qquad \times k\left(s - s' + \frac{1}{2}(xy' - yx'), x - x', y - y'\right) \mathrm{d}s\, \mathrm{d}x'\, \mathrm{d}y' \\
&= c_h^n \int_{\mathbb{R}^{2n}} \mathrm{e}^{\pi \mathrm{i} h(xy'-yx')} \hat{k}'_s(h, x', y') \hat{k}_s(h, x - x', y - y')\, \mathrm{d}x'\, \mathrm{d}y'.
\end{aligned} \tag{3.5}$$

Note that if we apply the Fourier transform $(x, y) \to (q, p)$ to the last expression (3.5), we get the *star product* of $\hat{k}'$ and $\hat{k}$ known in *deformation* quantization, cf [36, (9)–(13)]. Consequently, the representation $\rho_h([k', k])$ of the commutator (3.2) depends only on

$$\begin{aligned}
[k', k]\hat{}_s &= c_h^n \int_{\mathbb{R}^{2n}} (\mathrm{e}^{\mathrm{i}\pi h(xy'-yx')} - \mathrm{e}^{-\mathrm{i}\pi h(xy'-yx')}) \hat{k}'_s(-h, x', y') \hat{k}_s(-h, x - x', y - y')\, \mathrm{d}x'\, \mathrm{d}y' \\
&= 2\mathrm{i} c_h^n \int_{\mathbb{R}^{2n}} \sin(\pi h(xy' - yx')) \hat{k}'_s(h, x', y') \hat{k}_s(h, x - x', y - y')\, \mathrm{d}x'\, \mathrm{d}y'.
\end{aligned} \tag{3.6}$$

The integral (3.6) turns out to be equivalent to the *Moyal brackets* [36] for the (full) Fourier transforms of k' and k. It is commonly accepted that the method of orbit is a mathematical side of the *geometric* quantization [35]. Our derivation of the Moyal brackets in terms of orbits shows that deformation and geometric quantizations are closely connected and both are not very far from the original quantization of Heisenberg and Schrödinger. Yet one more of their close relatives can be identified as the Berezin quantization [4], see the next subsection.

Remark 3.2. Expression (3.6) vanishes for $h = 0$ as can be expected from the commutativity of representations (2.15). Thus it does not produce anything interesting on $\mathcal{O}_0$, which supports the common negligence of this set.

Summing up, p-mechanical observables, i.e. convolutions on $L_2(\mathbb{H}^n)$, are transformed

(1) by representations ρ_h (2.9) into quantum observables (3.3) with the Moyal bracket (3.6) between them;

(2) by representations $\rho_{(q,p)}$ (2.15) into classical observables (3.4).

We did not get meaningful brackets on classical observables yet; this will be done in section 4.1.

3.2. Berezin quantization and wavelet transform

There is the following construction, known as the *Berezin quantization* [3, 4], allowing us to assign a function to an operator (observable) and an operator to a function. The scheme is based on the construction of the *coherent states* and can be derived from different sources [29, 30]. We prefer the group-theoretic origin of Perelomov coherent states [30], which is realized in (2.21). Following [3] we introduce the *covariant* symbol $a(g)$ of an operator A on $F_2(\mathcal{O}_h)$ by the simple expression

$$a(g) = \langle A v_g, v_g \rangle \tag{3.7}$$

i.e. we get a map from the linear space of operators on $F_2(\mathcal{O}_h)$ to a linear space of functions on $\mathbb{H}^n$. A map in the opposite direction assigns to a function $\breve{a}(g)$ on $\mathbb{H}^n$ the linear operator A on $F_2(\mathcal{O}_h)$ by the formula

$$A = c_h^{n+1} \int_{\mathbb{H}^n} \mathring{a}(g) P_g \,\mathrm{d}g \qquad \text{where } P_g \text{ is the projection } P_g v = \langle v, v_g \rangle v_g. \tag{3.8}$$

The function $\mathring{a}(g)$ is called the *contravariant* symbol of the operator A (3.8).

The co- and contravariant symbols of operators are defined through the coherent states; in fact both types of symbols are realizations [25, section 3.1] of the direct (2.23) and inverse (2.22) wavelet transforms. Let us define a representation ρ_{bh} of the group $\mathbb{H}^n \times \mathbb{H}^n$ in the space $\mathcal{B}(F_2(\mathcal{O}_h))$ of operators on $F_2(\mathcal{O}_h)$ by the formula

$$\rho_{bh}(g_1, g_2) : A \mapsto \rho_h\left(g_1^{-1}\right) A \rho_h(g_2) \qquad \text{where} \quad g_1, g_2 \in \mathbb{H}^n. \tag{3.9}$$

According to the scheme from [25] for any state f_0 on $\mathcal{B}(F_2(\mathcal{O}_h))$, we get the wavelet transform $\mathcal{W}_{f_0} : \mathcal{B}(F_2(\mathcal{O}_h)) \to C(\mathbb{H}^n \times \mathbb{H}^n)$:

$$\mathcal{W}_{f_0} : A \mapsto \breve{a}(g_1, g_2) = \langle \rho_{bh}(g_1, g_2) A, f_0 \rangle. \tag{3.10}$$

The important particular case is given by f_0 defined through the vacuum vector v_0 (2.18) by the formula $\langle A, f_0 \rangle_{\mathcal{B}(F_2(\mathcal{O}_h))} = \langle A v_0, v_0 \rangle_{F_2(\mathcal{O}_h)}$. Then the wavelet transform (3.10) produces the *covariant presymbol* $\breve{a}(g_1, g_2)$ of operator A. Its restriction $a(g) = \breve{a}(g, g)$ to the diagonal D of $\mathbb{H}^n \times \mathbb{H}^n$ is exactly [25] the Berezin covariant symbol (3.7) of A. Such a restriction to the diagonal is done without loss of information due to holomorphic properties of $\breve{a}(g_1, g_2)$ [3].

Another important example of the state f_0 is given by the trace

$$\langle A, f_0 \rangle = \operatorname{Tr} A = h^n \int_{\mathbb{R}^{2n}} \langle A v_{(x,y)}, v_{(x,y)} \rangle_{F_2(\mathcal{O}_h)} \,\mathrm{d}x \,\mathrm{d}y \tag{3.11}$$

where coherent states $v_{(x,y)}$ are again defined in (2.21). Operators $\rho_{bh}(g, g)$ from the diagonal D of $\mathbb{H}^n \times \mathbb{H}^n$ trivially act on the wavelet transform (3.10) generated by the trace (3.11) since the trace is invariant under $\rho_{bh}(g, g)$. According to the general scheme [25] we can consider *reduced wavelet transform* to the homogeneous space $\mathbb{H}^n \times \mathbb{H}^n / D$ instead of the entire group $\mathbb{H}^n \times \mathbb{H}^n$. The space $\mathbb{H}^n \times \mathbb{H}^n / D$ is isomorphic to $\mathbb{H}^n$ with the embedding $\mathbb{H}^n \to \mathbb{H}^n \times \mathbb{H}^n$ given by $g \mapsto (g; 0)$. Furthermore the centre Z of $\mathbb{H}^n$ acts trivially in the representation ρ_{bh} as usual. Thus the only essential part of $\mathbb{H}^n \times \mathbb{H}^n / D$ in the wavelet transform is the homogeneous space $\Omega = \mathbb{H}^n / Z$. A Borel section $\mathbf{s} : \Omega \to \mathbb{H}^n \times \mathbb{H}^n$ in the principal bundle $G \to \Omega$ can be defined as $\mathbf{s} : (x, y) \mapsto ((0, x, y); (0, 0, 0))$. We got the reduced realization $\mathcal{W}_r$ of the wavelet transform (3.10) of the form

$$\begin{aligned}
\mathcal{W}_r : A \mapsto \breve{a}_r(x, y) &= \langle \rho_{bh}(\mathbf{s}(x, y))A, f_0 \rangle \\
&= \mathrm{Tr}(\rho_h((0, x, y)^{-1})A) \qquad (3.12)\\
&= h^n \int_{\mathbb{R}^{2n}} \langle \rho_h((0, x, y)^{-1})Av_{(x',y')}, v_{(x',y')}\rangle_{F_2(\mathcal{O}_h)}\, \mathrm{d}x'\, \mathrm{d}y' \\
&= h^n \int_{\mathbb{R}^{2n}} \langle Av_{(x',y')}, v_{(x,y)\cdot(x',y')}\rangle_{F_2(\mathcal{O}_h)}\, \mathrm{d}x'\, \mathrm{d}y'. \qquad (3.13)
\end{aligned}$$

Formula (3.12) is the principal ingredient of the *inversion formula* for the Heisenberg group [11, chapter 1, (1.60)] and [33, chapter 1, theorem 2.7] , which reconstructs kernels of convolutions $k(g)$ out of operators $\rho_h(k)$. Therefore if we define a mother wavelet to be the identity operator I, the inverse wavelet transform (cf (2.22) will be

$$\begin{aligned}
\mathcal{M}_r a &= h^n \int_{\mathbb{R}^{2n}} a(x, y)\rho_{bh}(\mathbf{s}((0, x, y)^{-1}))I \,\mathrm{d}x\, \mathrm{d}y \\
&= h^n \int_{\mathbb{R}^{2n}} a(x, y)\rho_h(0, x, y)\, \mathrm{d}x\, \mathrm{d}y. \qquad (3.14)
\end{aligned}$$

The inversion formula for $\mathbb{H}^n$ ensures the following proposition.

Proposition 3.3. *The composition $\mathcal{M}_r \circ \mathcal{W}_r$ is the identity map on the representations $\rho_h(k)$ of convolution operators on $\mathcal{O}_h$.*

Example 3.4. The wavelet transform $\mathcal{W}_r$ (3.13) applied to the quantum coordinate $Q = \mathrm{d}\rho_h(X)$, momentum $P = \mathrm{d}\rho_h(Y)$ (see (2.11)) and the energy function of the harmonic oscillator $(c_1 Q^2 + c_2 P^2)/2$ produces the distributions on $\mathbb{R}^{2n}$:

$$\begin{aligned}
Q &\mapsto \frac{1}{2\pi \mathrm{i}}\delta^{(1)}(x)\delta(y) \\
P &\mapsto \frac{1}{2\pi \mathrm{i}}\delta(x)\delta^{(1)}(y) \\
\frac{1}{2}(c_1 Q^2 + c_2 P^2) &\mapsto -\frac{1}{8\pi^2}(c_1\delta^{(2)}(x)\delta(y) + c_2\delta(x)\delta^{(2)}(y))
\end{aligned}$$

where $\delta^{(1)}$ and $\delta^{(2)}$ are the first and second derivatives of the Dirac *delta function* δ respectively. The constants c_1 and c_2 have units M/T^2 and $1/M$ correspondingly. We will use these distributions later in example 3.7.

3.3. From classical and quantum observables to p-mechanics

It is commonly accepted that we cannot deal with quantum mechanics directly and thus classical dynamics serves as an unavoidable intermediate step. A passage from classical observables to quantum ones—known as *quantization*—is a huge field with many concurring approaches (geometric, deformation, Weyl, Berezin, etc quantizations) each having its own merits and demerits. Similarly one has to construct *p*-mechanical observables starting from classical or quantum ones by some procedure (should it be named '*p*-mechanization'?), which we are about to describe.

The transition from a *p*-mechanical observable to the classical one is given by formula (3.4), which in turn is a realization of the inverse wavelet transform (2.22):

$$\rho_{(q,p)}k = \hat{k}(0, q, p) = c_h^{n+1} \int_{\mathbb{H}^n} k(s, x, y)\, \mathrm{e}^{-2\pi \mathrm{i}(qx+py)}\, \mathrm{d}s\, \mathrm{d}x\, \mathrm{d}y. \qquad (3.15)$$

Similar to the case of quantization, the classical image $\rho_{(q,p)}k$ (3.15) contains only partial information about p-observable k unless we make some additional assumptions. Let us start from a classical observable $c(q, p)$ and construct the corresponding p-observable. From the general consideration (see [25] and section 2.3) we can partially invert formula (3.15) by the wavelet transform (2.23):

$$\check{c}(x, y) = [\mathcal{W}_0 c](x, y) = \langle c v_{(0,0)}, v_{(x,y)}\rangle = c_h^n \int_{\mathbb{R}^{2n}} c(q, p)\, \mathrm{e}^{2\pi \mathrm{i}(qx+py)}\, \mathrm{d}q\, \mathrm{d}p \tag{3.16}$$

where $v_{(x,y)} = \rho_{(q,p)} v_{(0,0)} = \mathrm{e}^{-2\pi \mathrm{i}(qx+py)}$.

However, the function $\check{c}(x, y)$ (3.16) is not defined on the entire $\mathbb{H}^n$. The natural domain of $\check{c}(x, y)$ according to the construction of the reduced wavelet transform [25] is the homogeneous space $\Omega = G/Z$, where $G = \mathbb{H}^n$ and Z is its normal subgroup of central elements $(s, 0, 0)$. Let $\mathbf{s} : \Omega \to G$ be a Borel section in the principal bundle $G \to \Omega$, which is used in the construction of the induced representation, see [19, section 13.1]. For the Heisenberg group [25, ex. 4.3] it can be simply defined as $\mathbf{s} : (x, y) \in \Omega \mapsto (0, x, y) \in \mathbb{H}^n$. One can naturally transfer functions from Ω to the image $\mathbf{s}(\Omega)$ of the map $\mathbf{s}$ in G. However, the range $\mathbf{s}(\Omega)$ of $\mathbf{s}$ often has (particularly for $\mathbb{H}^n$) a zero Haar measure in G. Probably the two simplest possible ways out are:

(1) To increase the 'weight' of function $\tilde{c}(s, x, y)$ vanishing outside the range $\mathbf{s}(\Omega)$ of $\mathbf{s}$ by a suitable Dirac delta function on the subgroup Z. For the Heisenberg group this can be done, for example, by the map

$$\mathcal{E} : \check{c}(x, y) \mapsto \tilde{c}(s, x, y) = \delta(s)\check{c}(x, y) \tag{3.17}$$

where $\check{c}(x, y)$ is given by the inverse wavelet (Fourier) transform (3.16). As we will see in proposition 3.6 this is related to the Weyl quantization and the Moyal brackets.

(2) To extend the function $\check{c}(x, y)$ to the entire group G by a tensor product with a suitable function on Z, for example e^{-s^2}:

$$\check{c}(x, y) \mapsto \tilde{c}(s, x, y) = \mathrm{e}^{-s^2}\check{c}(x, y).$$

In order to get the *correspondence principle* between classical and quantum mechanics (cf example 2.3), the function on Z has to satisfy some additional requirements. For $\mathbb{H}^n$ it should vanish for $s \to \pm\infty$, which is fulfilled for both e^{-s^2} and $\delta(s)$ from the previous item. In this way we get infinitely many essentially different quantizations with non-equivalent *deformed* Moyal brackets between observables.

There are other more complicated possibilities not mentioned here, which can be of some use if additional information or assumptions are used to extend functions from Ω to G. We will focus here only on the first 'minimalistic' approach from the two listed above.

Example 3.5. The composition of the wavelet transform $\mathcal{W}_0$ (3.16) and the map $\mathcal{E}$ (3.17) applied to the classical coordinate, momentum and the energy function of a harmonic oscillator produces the following distributions on $\mathbb{H}^n$:

$$q \mapsto \frac{1}{2\pi \mathrm{i}}\delta(s)\delta^{(1)}(x)\delta(y) \tag{3.18}$$

$$p \mapsto \frac{1}{2\pi \mathrm{i}}\delta(s)\delta(x)\delta^{(1)}(y) \tag{3.19}$$

$$\frac{1}{2}(c_1 q^2 + c_2 p^2) \mapsto -\frac{1}{8\pi^2}(c_1\delta(s)\delta^{(2)}(x)\delta(y) + c_2\delta(s)\delta(x)\delta^{(2)}(y)) \tag{3.20}$$

where $\delta^{(1)}$, $\delta^{(2)}$, c_1 and c_2 are defined in example 3.4. We will use these distributions later in the example 4.3.

If we apply the representation ρ_h (3.3) to the function $\tilde{c}(s, x, y)$ (3.17) we will get the operator on $F_2(\mathcal{O}_h)$:

$$\begin{aligned}
\mathcal{Q}_h(c) &= c_h^{n+1} \int_{\mathbb{H}^n} \tilde{c}(s, x, y)\rho_h(s, x, y)\, \mathrm{d}s\, \mathrm{d}x\, \mathrm{d}y \\
&= c_h^{n+1} \int_{\mathbb{R}^{2n}} \int_{\mathbb{R}} \delta(s)\check{c}(x, y) \exp(s\, \mathrm{d}\rho_h(S) + x\, \mathrm{d}\rho_h(X) + y\, \mathrm{d}\rho_h(Y))\, \mathrm{d}s\, \mathrm{d}x\, \mathrm{d}y \\
&= c_h^{n+1} \int_{\mathbb{R}} \delta(s)\, \mathrm{e}^{-2\pi \mathrm{i} s h}\, \mathrm{d}s \int_{\mathbb{R}^{2n}} \check{c}(x, y) \exp(x\, \mathrm{d}\rho_h(X) + y\, \mathrm{d}\rho_h(Y))\, \mathrm{d}x\, \mathrm{d}y \\
&= c_h^{n} \int_{\mathbb{R}^{2n}} \check{c}(x, y) \exp(x\, \mathrm{d}\rho_h(X) + y\, \mathrm{d}\rho_h(Y))\, \mathrm{d}x\, \mathrm{d}y
\end{aligned} \tag{3.21}$$

where the last expression is exactly the *Weyl quantization* (the *Weyl correspondence* [11, section 2.1]) if the Schrödinger realization with $\mathrm{d}\rho_h(X) = q$ and $\mathrm{d}\rho_h(Y) = \mathrm{i}h\partial_q$ on $L_2(\mathbb{R}^n)$ is chosen for ρ_h. Thus we demonstrate the following proposition.

Proposition 3.6. *The Weyl quantization $\mathcal{Q}_h$ (3.21) is the composition of the wavelet transform (3.16), the extension $\mathcal{E}$ (3.17) and the representation ρ_h (2.9):*

$$\mathcal{Q}_h = \rho_h \circ \mathcal{E} \circ \mathcal{W}_0. \tag{3.22}$$

A similar construction can be carried out if we have a quantum observable A and wish to recover a related p-mechanical object. The wavelet transform $\mathcal{W}_r$ (3.12) maps A into the function $a(x, y)$ defined on Ω and we again face the problem of extension of $a(x, y)$ to the entire group $\mathbb{H}^n$. If it can once more be solved as in the classical case by the tensor product with the delta function $\delta(s)$, then we get the following formula:

$$A \mapsto a(s, x, y) = \mathcal{E} \circ \mathcal{W}_r(A) = h^n \delta(s) \int_{\mathbb{R}^{2n}} \langle A v_{(x',y')}, v_{(x,y)\cdot(x',y')}\rangle_{F_2(\mathcal{O}_h)}\, \mathrm{d}x'\, \mathrm{d}y'.$$

We can apply to this function $a(s, x, y)$ the representation $\rho_{(q,p)}$ and obtain classical observables $\rho_{(q,p)}(a)$. For a reasonable quantum observable A, its classical image $\rho_{(q,p)} \circ \mathcal{E} \circ \mathcal{W}_r(A)$ will coincide with its classical limit $C_{h\to 0}A$:

$$C_{h\to 0} = \rho_{(q,p)} \circ \mathcal{E} \circ \mathcal{W}_r \tag{3.23}$$

which is expressed here through integral transformations and does not explicitly use any limit transition for $h \to 0$. Figure 2 illustrates various transformations between quantum, classical and p-observables. Besides the mentioned decompositions (3.22) and (3.23) there are presentations of identity maps on classical and quantum spaces correspondingly:

$$\mathcal{I}_c = \rho_{(q,p)} \circ \mathcal{E} \circ \mathcal{W}_0 \qquad \mathcal{I}_h = \rho_h \circ \mathcal{E} \circ \mathcal{W}_h.$$

Example 3.7. The wavelet transform $\mathcal{W}_r$ applied to the quantum coordinate Q, momentum P and the energy function of a harmonic oscillator $(c_1 Q^2 + c_2 P^2)/2$ was calculated in example 3.4. The composition with the above map $\mathcal{E}$ yields the following distributions:

$$\begin{aligned}
Q &\mapsto \frac{1}{2\pi \mathrm{i}} \delta(s)\delta^{(1)}(x)\delta(y) \\
P &\mapsto \frac{1}{2\pi \mathrm{i}} \delta(s)\delta^{(1)}(x)\delta(y) \\
\frac{1}{2}(c_1 Q^2 + c_2 P^2) &\mapsto -\frac{1}{2\pi^2}(c_1 \delta(s)\delta^{(2)}(x)\delta(y) + c_2 \delta(s)\delta(x)\delta^{(2)}(y))
\end{aligned}$$

which are exactly the same as in example 3.5.

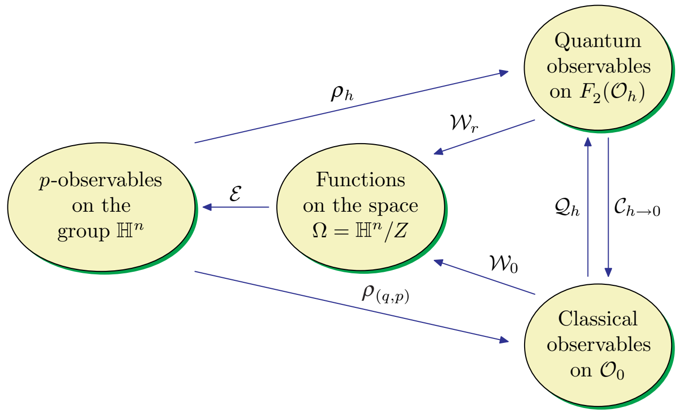

Figure 2. The relations between $\mathcal{Q}_h$ (the Weyl quantization from classical mechanics to quantum); $\mathcal{C}_{h\to 0}$ (the classical limit $h \to 0$ of quantum mechanics); ρ_h and $\rho_{(q,p)}$ (unitary representations of Heisenberg group $\mathbb{H}^n$); $\mathcal{W}_r$ and $\mathcal{W}_0$ (wavelet transforms defined in (3.12) and (3.16)) and $\mathcal{E}$ (extension of functions from $\Omega = \mathbb{H}^n/Z$ to the whole group $\mathbb{H}^n$). Note the relations $\mathcal{Q}_h = \rho_h \circ \mathcal{E} \circ \mathcal{W}_0$ and $C_{h\to 0} = \rho_{(q,p)} \circ \mathcal{E} \circ \mathcal{W}_r$.

4. *p*-mechanics: dynamics

We introduce the p-mechanical brackets, which suit all essential physical requirements and have a non-trivial classical representation coinciding with the Poisson brackets. A consistent p-mechanical dynamic equation is given in subsection 4.2 and is analysed for the harmonic oscillator. Symplectic automorphisms of the Heisenberg group produce symplectic symmetries of p-mechanical, quantum and classical dynamics in subsection 4.3.

4.1. p-mechanical brackets on $\mathbb{H}^n$

Having observables as convolutions on $\mathbb{H}^n$, we need a dynamic equation for their time evolution. To this end we seek a time derivative generated by an observable associated with energy.

Remark 4.1. The first candidate is the derivation obtained from commutator (3.2). However, the straight commutator suffers from at least two failures:

(1) It cannot produce any dynamics on $\mathcal{O}_0$ (2.17), see remark 3.2.
(2) It violates convention 2.1 as indicated below.

As is well known the classical energy is measured in ML^2/T^2 and so is the p-mechanical energy E. Consequently, the commutator $[E, \cdot]$ (3.2) with the p-energy has units ML^2/T^2 whereas the time derivative should be measured in $1/T$, i.e. the mismatch is in units of action ML^2/T.

Fortunately, there is a possibility of fixing both the above defects of the straight commutator at the same time. Let us define a multiple $\mathcal{A}$ of a right inverse operator to the vector field S (2.3) on $\mathbb{H}^n$ by its actions on exponents—characters of the centre $Z \in \mathbb{H}^n$:

$$S\mathcal{A} = 4\pi^2 I \qquad \text{where} \quad \mathcal{A}\,\mathrm{e}^{2\pi\mathrm{i}hs} = \begin{cases} \frac{2\pi}{\mathrm{i}h}\,\mathrm{e}^{2\pi\mathrm{i}hs} & \text{if} \quad h \neq 0 \\ 4\pi^2 s & \text{if} \quad h = 0. \end{cases} \tag{4.1}$$

An alternative definition of $\mathcal{A}$ as the convolution with a distribution is given in [27].

We can extend $\mathcal{A}$ by the linearity to the entire space $L_1(\mathbb{H}^n)$. As a multiplier of a right inverse to S, the operator $\mathcal{A}$ is measured in $T/(ML^2)$—exactly what we need to correct the second of the above mentioned defects of the straight commutator. Thus we introduce [27] the modified convolution operation $\star$ on $L_1(\mathbb{H}^n)$:

$$k' \star k = (k' * k)\mathcal{A} \tag{4.2}$$

and the associated modified commutator (p-mechanical brackets):

$$[\![k', k]\!] = [k', k]\mathcal{A} = k' \star k - k \star k'. \tag{4.3}$$

Obviously (4.3) is a bilinear antisymmetric form on the convolution kernels. It was also demonstrated in [27] that p-mechanical brackets satisfy the Leibniz and Jacobi identities. They are all important for a consistent dynamics [8] along with the dimensionality condition given in the beginning of this subsection.

From (3.3) one gets $\rho_h(\mathcal{A}k) = \frac{2\pi}{\mathrm{i}h}\rho_h(k)$ for $h \neq 0$. Consequently the modification of the commutator for $h \neq 0$ is only slightly different from the original one:

$$\rho_h[\![k', k]\!] = \frac{1}{\mathrm{i}\hbar}[\rho_h(k'), \rho_h(k)] \qquad \text{where} \quad \hbar = \frac{h}{2\pi} \neq 0. \tag{4.4}$$

The integral representation of the modified commutator kernel becomes (cf (3.6))

$$\hat{[\![k', k]\!]}_s = c_h^n \int_{\mathbb{R}^{2n}} \frac{4\pi}{h} \sin(\pi h(xy' - yx'))\hat{k}'_s(h, x', y')\hat{k}_s(h, x - x', y - y')\,\mathrm{d}x'\,\mathrm{d}y' \tag{4.5}$$

where we can understand the expression under the integral as

$$\frac{4\pi}{h}\sin(\pi h(xy' - yx')) = 4\pi^2 \sum_{k=1}^{\infty}(-1)^{k+1}(\pi h)^{2(k-1)}\frac{(xy' - yx')^{2k-1}}{(2k-1)!}. \tag{4.6}$$

This makes the operation (4.5) for $h = 0$ significantly distinct from the vanishing integral (3.6). Indeed, it is natural to assign the value $4\pi^2(xy' - yx')$ to (4.6) for $h = 0$. Then the integral in (4.5) becomes the Poisson brackets for the Fourier transforms of k' and k defined on $\mathcal{O}_0$ (2.17):

$$\rho_{(q,p)}[\![k', k]\!] = \frac{\partial \hat{k}'(0, q, p)}{\partial q}\frac{\partial \hat{k}(0, q, p)}{\partial p} - \frac{\partial \hat{k}'(0, q, p)}{\partial p}\frac{\partial \hat{k}(0, q, p)}{\partial q}. \tag{4.7}$$

The same formula is obtained [27, proposition 3.5] if we directly calculate $\rho_{(q,p)}[\![k', k]\!]$ rather than resolve the indeterminacy for $h = 0$ in (4.6). This means the continuity of our construction at $h = 0$ and represents the *correspondence principle* between quantum and classical mechanics.

We saw that the remedy of the second failure of commutator in remark 4.1 (which was our duty according to convention 2.1) by the anti-derivative (4.1) also improves the first defect (which is a very pleasant and surprising bonus). There are probably much simpler ways to fix the dimensionality of commutator 'by hand'. However, not all of them obviously would produce the Poisson brackets on $\mathcal{O}_0$ as the anti-derivative (4.1).

We arrived at the following observation: *Poisson brackets and inverse of the Planck constant $1/h$ have the same dimensionality because they are images of the same object (anti-derivative (4.1)) under different representations (2.9) and (2.15) of the Heisenberg group.*

Note that functions $X = \delta(s)\delta^{(1)}(x)\delta(y)$ and $Y = \delta(s)\delta(x)\delta^{(1)}(y)$ (see (3.18) and (3.19)) on $\mathbb{H}^n$ are measured in units of L and ML^2/T (inverse to x and y) correspondingly because they are respective derivatives of the dimensionless function $\delta(s)\delta(x)\delta(y)$. Then the p-mechanical brackets $[\![X, \cdot]\!]$ and $[\![Y, \cdot]\!]$ with these functions have dimensionality of $T/(ML^2)$ and

$1/L$ correspondingly. Their representations $\rho_*[\![X, \cdot]\!]$ and $\rho_*[\![Y, \cdot]\!]$ (for both types of representations ρ_h and $\rho_{(q,p)}$) are measured by L and ML^2/T and are simple derivatives:

$$\rho_*[\![X, \cdot]\!] = \frac{\partial}{\partial p} \qquad \rho_*[\![Y, \cdot]\!] = \frac{\partial}{\partial q}. \tag{4.8}$$

Thus $\rho_*[\![X, \cdot]\!]$ and $\rho_*[\![Y, \cdot]\!]$ are generators of shifts on both types of orbits $\mathcal{O}_h$ and $\mathcal{O}_0$ independent of the value of h.

4.2. p-mechanical dynamic equation

Since the modified commutator (4.3) with a p-mechanical energy has the dimensionality $1/T$—the same as the time derivative—we introduce the dynamic equation for an observable $f(s, x, y)$ on $\mathbb{H}^n$ based on that modified commutator as follows:

$$\frac{\mathrm{d}f}{\mathrm{d}t} = [\![f, E]\!]. \tag{4.9}$$

Remark 4.2. It is a general tendency to make Poisson brackets or a quantum commutator out of any two observables and say that they form a Lie algebra. However, there is a physical meaning to do that if at least one of two observables is an energy, coordinate or momentum: in these cases the brackets produce the time derivative (4.9) or corresponding shift generators (4.8) [16] of the other observable.

A simple consequence of the previous consideration is that the p-dynamic equation (4.9) is reduced

(1) by the representation ρ_h, $h \neq 0$ (2.9) on $F_2(\mathcal{O}_h)$ (2.7) to Moyal's form of Heisenberg equation [36, (8)] based on the formulae (4.4) and (4.5):

$$\frac{\mathrm{d}\rho_h(f)}{\mathrm{d}t} = \frac{1}{\mathrm{i}\hbar}[\rho_h(f), H_h] \qquad \text{where the operator } H_h = \rho_h(E) \tag{4.10}$$

(2) by the representations $\rho_{(q,p)}$ (2.15) on $\mathcal{O}_0$ (2.17) to Poisson's equation [2, section 39] based on the formula (4.7):

$$\frac{\mathrm{d}\hat{f}}{\mathrm{d}t} = \{\hat{f}, H\} \qquad \text{where the function } H(q, p) = \rho_{(q,p)}E = \hat{E}(0, q, p). \tag{4.11}$$

The same connections are true for the solutions of the three equations (4.9)–(4.11).

Example 4.3 (harmonic oscillator, of course) [27]**.** Let the p-mechanical energy function of a harmonic oscillator be as those obtained in examples 3.5 and 3.7:

$$E(s, x, y) = -\frac{1}{8\pi^2}(c_1\delta(s)\delta^{(2)}(x)\delta(y) + c_2\delta(s)\delta(x)\delta^{(2)}(y)). \tag{4.12}$$

Then the p-dynamic equation (4.9) on $\mathbb{H}^n$ obeying convention 2.1 is

$$\frac{\mathrm{d}}{\mathrm{d}t} f(t; s, x, y) = \sum_{j=1}^{n} \left(c_2 x_j \frac{\partial}{\partial y_j} - c_1 y_j \frac{\partial}{\partial x_j} \right) f(t; s, x, y). \tag{4.13}$$

Solutions to the above equation are well known to be rotations in each of (x_j, y_j) planes given by

$$f(t; s, x, y) = f_0 \left(s, x \cos(\sqrt{c_1 c_2} t) - \sqrt{\frac{c_1}{c_2}} y \sin(\sqrt{c_1 c_2} t), \right.$$
$$\left. \times \sqrt{\frac{c_2}{c_1}} x \sin(\sqrt{c_1 c_2} t) + y \cos(\sqrt{c_1 c_2} t) \right). \tag{4.14}$$

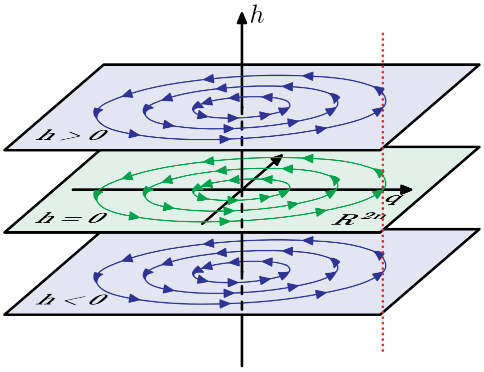

Figure 3. Dynamics of the harmonic oscillator in the adjoint space $\mathfrak{h}_n^*$ is given by the identical linear symplectomorphisms of all orbits $\mathcal{O}_h$ and $\mathcal{O}_0$. The vertical dotted string is uniformly rotating in the 'horizontal' plane around the h-axis without any dynamics along the 'vertical' direction.

This expression respects convention 2.1. Since the dynamics on $L_2(\mathbb{H}^n)$ is given by a symplectic linear transformation of $\mathbb{H}^n$, its Fourier transform (2.10) to $L_2(\mathfrak{h}_n^*)$ is the adjoint symplectic linear transformation of orbits $\mathcal{O}_h$ and $\mathcal{O}_0$ in $\mathfrak{h}_n^*$, see figure 3.

The representation ρ_h transforms the energy function E (4.12) into the operator

$$H_h = -\frac{1}{8\pi^2}(c_1 Q^2 + c_2 P^2) \tag{4.15}$$

where $Q = \mathrm{d}\rho_h(X)$ and $P = \mathrm{d}\rho_h(Y)$ are defined in (2.11). The representation $\rho_{(q,p)}$ transforms E into the classical Hamiltonian

$$H(q,p) = \frac{c_1}{2}q^2 + \frac{c_2}{2}p^2. \tag{4.16}$$

The p-dynamic equation (4.9) of form (4.13) is transformed by the representations ρ_h into the Heisenberg equation

$$\frac{\mathrm{d}}{\mathrm{d}t}f(t;Q,P) = \frac{1}{\mathrm{i}\hbar}[f,H_h] \qquad \text{where} \qquad \frac{1}{\mathrm{i}\hbar}[f,H_h] = c_1 p\frac{\partial f}{\partial q} - c_2 q\frac{\partial f}{\partial p} \tag{4.17}$$

defined by the operator H_h (4.15). The representation $\rho_{(q,p)}$ produces the Hamilton equation

$$\frac{\mathrm{d}}{\mathrm{d}t}f(t;q,p) = c_1 p\frac{\partial f}{\partial q} - c_2 q\frac{\partial f}{\partial p} \tag{4.18}$$

defined by the Hamiltonian $H(q,p)$ (4.16). Finally, to get the solution for equations (4.17) and (4.18) it is enough to apply representations ρ_h and $\rho_{(q,p)}$ to the solution (4.14) of p-dynamic equation (4.13).

Summing up we can rephrase the title of [36]: *quantum and classical mechanics live and work together on the Heisenberg group and are separated only in irreducible representations of* $\mathbb{H}^n$.

4.3. Symplectic invariance from automorphisms of $\mathbb{H}^n$

Let $A : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be a linear *symplectomorphism* [2, section 41] and [11, section 4.1], i.e. a map defined by $2n \times 2n$ matrix:

$$A : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix}$$

preserving the symplectic form (2.2):

$$\omega(A(x, y); A(x', y')) = \omega(x, y; x', y'). \tag{4.19}$$

All such transformations form the symplectic group $Sp(n)$. Convention 2.1 implies that sub-blocks a and d of A have to be dimensionless while b and c have to be of reciprocal dimensions M/T and T/M respectively.

It follows from identities (4.19) and (2.1) that the linear transformation $\alpha : \mathbb{H}^n \to \mathbb{H}^n$ such that $\alpha(s, x, y) = (s, A(x, y))$ is an automorphism of $\mathbb{H}^n$. Let us also denote by $\tilde{\alpha} = \tilde{\alpha}_A$ a unitary transformation of $L_2(\mathbb{H}^n)$ in the form

$$\tilde{\alpha}(f)(s, x, y) = \sqrt{\det a}\, f(s, A(x, y))$$

which is well defined [11, section 4.2] on the double cover $\widetilde{Sp}(n)$ of the group $Sp(n)$. The correspondence $A \mapsto \tilde{\alpha}_A$ is a linear unitary representation of the symplectic group in $L_2(\mathbb{H}^n)$. One can also check the intertwining property

$$\lambda_{l(r)}(g) \circ \tilde{\alpha} = \tilde{\alpha} \circ \lambda_{l(r)}(\alpha(g)) \tag{4.20}$$

for the left (right) regular representations (2.5) of $\mathbb{H}^n$.

Because α is an automorphism of $\mathbb{H}^n$ the map $\alpha^* : k(g) \mapsto k(\alpha(g))$ is an automorphism of the convolution algebra $L_1(\mathbb{H}^n)$ with the multiplication $*$ (3.1), i.e. $\alpha^*(k_1) * \alpha^*(k_2) = \alpha^*(k_1 * k_2)$. Moreover, α^* commutes with the anti-derivative $\mathcal{A}$ (4.1), thus α^* is also an automorphism of $L_1(\mathbb{H}^n)$ with the modified multiplication $\star$ (4.2), i.e. $\alpha^*(k_1) \star \alpha^*(k_2) = \alpha^*(k_1 \star k_2)$. By linearity, we can extend the intertwining property (4.20) to the convolution operator K as follows:

$$\alpha^* K \circ \tilde{\alpha} = \tilde{\alpha} \circ K. \tag{4.21}$$

Since α is an automorphism of $\mathbb{H}^n$ it fixes the unit e of $\mathbb{H}^n$ and its differential $\mathrm{d}\alpha : \mathfrak{h}^n \to \mathfrak{h}^n$ at e is given by the same matrix as α in the exponential coordinates. Obviously, $\mathrm{d}\alpha$ is an automorphism of the Lie algebra $\mathfrak{h}^n$. By the duality between $\mathfrak{h}^n$ and $\mathfrak{h}^*_n$ we obtain the adjoint map $\mathrm{d}\alpha^* : \mathfrak{h}^*_n \to \mathfrak{h}^*_n$ defined by the expression

$$\mathrm{d}\alpha^* : (h, q, p) \mapsto (h, A^t(q, p)) \tag{4.22}$$

where A^t is the transpose of A. Obviously, $\mathrm{d}\alpha^*$ preserves any orbit $\mathcal{O}_h$ (2.7) and maps the orbit $\mathcal{O}_{(q,p)}$ (2.8) to $\mathcal{O}_{A^t(q,p)}$.

Identity (4.22) indicates that both representations ρ_h and $(\rho_h \circ \alpha)(s, x, y) = \rho_h(s, A(x, y))$ for $h \neq 0$ correspond to the same orbit $\mathcal{O}_h$. Thus they should be equivalent, i.e. there is an intertwining operator $U_A : F_2(\mathcal{O}_h) \to F_2(\mathcal{O}_h)$ such that $U_A^{-1} \rho_h U_A = \rho_h \circ \alpha$. Then the correspondence $\sigma : A \mapsto U_A$ is a linear unitary representation of the double cover $\widetilde{Sp}(n)$ of the symplectic group called the *metaplectic representation* [11, section 4.2] and [13].

Thus we have the following proposition.

Proposition 4.4. *The p-mechanical brackets are invariant under the symplectic automorphisms of* $\mathbb{H}^n$*:* $[\![\tilde{\alpha}k_1, \tilde{\alpha}k_2]\!] = \tilde{\alpha}[\![k_1, k_2]\!]$*. Consequently the dynamic equation (4.9) has symplectic symmetries which are reduced*

(1) by ρ_h, $h \neq 0$ *on* $\mathcal{O}_h$ *(2.7) to the metaplectic representation in quantum mechanics;*
(2) by $\rho_{(q,p)}$ *on* $\mathcal{O}_0$ *(2.17) to the symplectic symmetries of classical mechanics [2, section 38].*

Combining intertwining properties of all three components (3.22) in the Weyl quantization we get the following corollary.

Corollary 4.5. *The Weyl quantization* $\mathcal{Q}_h$ *(3.21) is the intertwining operator between classical and metaplectic representations.*

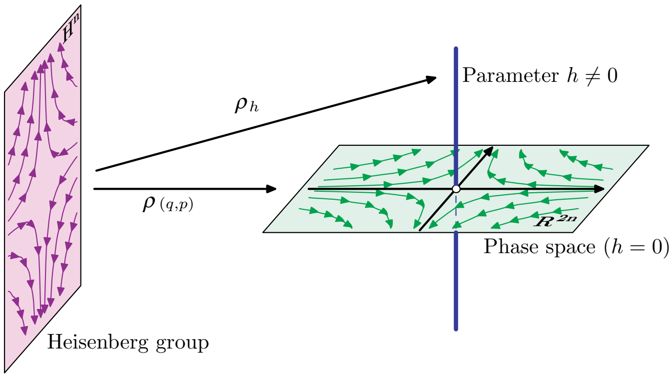

Figure 4. Automorphisms of $\mathbb{H}^n$ generated by the symplectic group $Sp(n)$ do not mix representations ρ_h with different Planck constants h and act by the metaplectic representation inside each of them. In contrast, those automorphisms of $\mathbb{H}^n$ act transitively on the set of one-dimensional representations $\rho_{(q,p)}$ joining them into the tangent space of the classical phase space $\mathbb{R}^{2n}$.

5. Conclusions

5.1. Discussion

Our intention is to demonstrate that the complete representation theory of the Heisenberg group $\mathbb{H}^n$, which includes one-dimensional commutative representations, is a sufficient language for *both* classical and quantum theory.

It is natural to describe the complete set of unitary irreducible representations by the orbit method of Kirillov. The analysis carried out in section 2.2 and illustrated in figure 1 shows that the position of one-dimensional representations $\rho_{(q,p)}$ within the unitary dual of $\mathbb{H}^n$ relates them to classical mechanics. Various connections of infinite dimensional representations ρ_h of $\mathbb{H}^n$ to quantum mechanics have been known for a long time.

Convolution operators on $\mathbb{H}^n$ are a natural class to be associated with physical observables. They are reduced by infinite dimensional representations ρ_h to the pseudodifferential operators, which are observables in the Weyl quantization. The one-dimensional representations $\rho_{(q,p)}$ map convolutions onto classical observables—functions on the phase space. The wavelet technique allows us to transform these three types of observables into each other, which is illustrated in figure 2.

A nontrivial dynamics in the phase space—the space of one-dimensional representations of $\mathbb{H}^n$—could be obtained from the commutator on $\mathbb{H}^n$ with the help of the anti-derivative operator $\mathcal{A}$ (4.1). The p-mechanical dynamic equation (4.9) based on the operator $\mathcal{A}$ possesses all desirable properties for the description of a physical time evolution and its solution gives both classical and quantum dynamics. See figure 3 for a familiar dynamics of the harmonic oscillator.

Finally, the symplectic automorphisms of the Heisenberg group preserve the dynamic equation (4.9) and all its solutions. In representations of the Heisenberg group this reduces to the symplectic invariance of classical mechanics and the metaplectic invariance of the quantum description. Moreover, the symplectic transformations act transitively on the set $\mathcal{O}_0$ (2.17) of one-dimensional representations supporting its p-mechanical interpretation as the classical phase space, see figure 4.

5.2. Further developments

The present paper deals only with elementary aspects of p-mechanics. The notion of physical states in p-mechanics is considered in [6, 7], where its usefulness for a forced oscillator is demonstrated. Paper [7] also discusses the connection of p-mechanics and *contextual interpretation* [18]. Our study is a part of the Erlangen-type approach [24, 26] in non-commutative geometry. It could be extended in several directions.

5.2.1. Quantum–classical interaction. The long standing discussion [8, 31] about quantum–classical interaction can be treated as follows. Let $\mathbb{B}$ be a nilpotent step two Heisenberg-like group of elements $(s_1, s_2; x_1, y_1; x_2, y_2)$ with the only non-trivial commutators in the Lie algebra (cf (2.4) as follows:

$$[X_i, Y_j] = \delta_{ij} S_i.$$

Thus $\mathbb{B}$ has the two-dimensional centre $(s_1, s_2, 0, 0, 0, 0)$ and the adjoint space of characters of $\mathbb{B}$ is also two dimensional. We can regard it as being spanned by two different Planck constants h_1 and h_2. There is a possibility of studying the case $h_1 \neq 0$ and $h_2 = 0$, which correspond to a quantum behaviour of coordinates (x_1, y_1) and a classical dynamics in (x_2, y_2). This study was initiated in [31] but oversaw some homological aspects of the construction and is not satisfactorily completed yet.

5.2.2. Quantum field theory. Mathematical formalism of quantum mechanics uses complex numbers in order to provide unitary infinite dimensional representations of the Heisenberg group $\mathbb{H}^n$. In a similar way the De Donder–Weyl formalism for classical field theories [17] requires *Clifford numbers* [14] for their quantization. It was recently realized [9] that the appearance of Clifford algebras is induced by the Galilean group—a nilpotent step two Lie group with multi-dimensional centre. In the one-dimensional case an element of the Galilean group is $(s_1, \dots, s_n, x, y_1, \dots, y_n)$ with corresponding Lie algebra described by the non-vanishing commutators

$$[X, Y_j] = S_j \qquad j = 1, 2, \dots, n.$$

This corresponds to several momenta $y_1, y_2, \dots, y_n$ adjoint to a single field coordinate x [17]. For field theories it is worth [28] considering Clifford valued representations induced by Clifford valued 'characters' $\exp(2\pi(e_1h_1s_1 + \dots + e_nh_ns_n))$ of the centre, where $e_1, \dots, e_n$ are imaginary units spanning the Clifford algebra. The associated Fock spaces were described in [9]. In [28] we quantize the De Donder–Weyl field equations (similar to our consideration in subsection 4.1) with the help of composed anti-derivative operator $\mathcal{A} = \sum_1^n e_i \mathcal{A}_i$, where $S_i\mathcal{A}_i = 4\pi^2 I$. There are important mathematical and physical questions related to the construction, notably the role of the Dirac operator [12], which deserve further careful consideration.

5.2.3. String theory. There is a possibility of using a p-mechanical picture for a string-like theory. Indeed the p-dynamics of a harmonic oscillator as presented in example 4.3 and figure 3 consists of uniform rotation of lines around the h-axis—one can say *strings*—with the same (q, p) coordinates but different values of the Planck constant h.

In the case of a more general energy, which is still however given by a *convolution* on $\mathbb{H}^n$, the dynamics can be more complicated. For example, it may not correspond to a point transformation of the adjoint space $\mathfrak{h}_n^*$. Alternatively, a generic point transformation may transform a straight line (h, q_0, p_0) with fixed $(q_0, p_0) \in \mathbb{R}^{2n}$ and variable h into a generic curve transversal to all (q, p)-planes. However, all spaces $F_2(\mathcal{O}_h)$ are invariant under any p-dynamics generated by a convolution on $\mathbb{H}^n$.

However, if energy is given by an arbitrary operator on $L_2(\mathbb{H}^n)$ [10, 22], spaces $F_2(\mathcal{O}_h)$ for different h are no longer invariant during the evolution and could be mixed together. This also opens a possibility of longitudinal dynamics of strings along the h-axis. It may seem strange to have a dynamics along h which is a *constant*, and not a variable. However, there is a duality [34] between the 'Planck constant' h and the 'tension of string' α'. Dualities and symmetries between h and α' can be reflected in dynamics which mixes spaces $F_2(\mathcal{O}_h)$ with different h.

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